



TITLE:

Relation between Equilibrium points for a Differential Inclusion and Solutions of a Variational Inequality(Information and mathematics of non-additivity and non-extensivity : from the viewpoint of functional analysis)

AUTHOR(S):

Tamura Manaka, Hiroko

CITATION:

Tamura Manaka, Hiroko. Relation between Equilibrium points for a Differential Inclusion and Solutions of a Variational Inequality(Information and mathematics of non-additivity and non-extensivity : from the viewpoint of functional analysis). 数理解 ...

ISSUE DATE:

2007-06

URL:

<http://hdl.handle.net/2433/81090>

RIGHT:

Relation between Equilibrium points for a Differential Inclusion and Solutions of a Variational Inequality

東京工業大学情報理工学 田村(真中)裕子 (Hiroko Manaka Tamura)
Department of Mathematical and Computing Science,
Tokyo Institute of Technology

We introduce a relation between equilibrium points for some differential inclusion and solutions of a variational inequality. At first we show convergence theorems of solutions for some differential inclusions.

1. Convergence Theorem for Differential Inclusions

Let X be a Hilbert space with an inner product (\cdot, \cdot) , let $K \subset X$ be a non-empty closed convex subset of X , let A be a set-valued mapping from K to 2^X with convex compact set-value, and let $x(\cdot) : [0, \infty) \rightarrow X$. We consider a differential inclusion $DI(A, K)$ for A and K as follows:

$$DI(A, K) : \begin{cases} x(t) \in K \text{ for any } t \in [0, \infty), \\ \dot{x}(t) \in -A(x(t)) \text{ for almost any } t \in (0, \infty). \end{cases}$$

Next we give some definitions.

定義 (Def.) (1) $x(\cdot)$ is called a trajectory of $DI(A, K)$, if a mapping $x(\cdot) : [0, \infty) \rightarrow K$ is absolutely continuous and satisfies $DI(A, K)$.

(2) A point $x^* \in K$ is said to be an equilibrium point for $DI(A, K)$ if $0 \in -A(x^*)$.

Let $\varphi : X \rightarrow (-\infty, \infty]$ be a proper lower semi-continuous convex function. A subdifferential $\partial\varphi : X \rightarrow 2^X$ is defined by

$$\partial\varphi(x) = \{w \in X : \varphi(y) \geq \varphi(x) + (w, y - x) \text{ for any } y \in X\}.$$

Then it is well-known that $x^* \in K$ is an equilibrium point of $DI(\partial\varphi, K)$ if and only if x^* is a minimum point of φ , i.e., $\varphi(x^*) = \min_{x \in X} \varphi(x)$. It is also known that $\partial\varphi$ has a property of *demipositivity*.

定義 (Def.) A set-valued mapping $A : X \rightarrow 2^X$ is said to be *demipositive* if (1), (2) and (3) hold:

(1) $(v, x - y) \geq 0$ for all $x \in X, y \in A^{-1}(0)$ and $v \in A(x)$.

(2) There exists $y_0 \in A^{-1}(0)$ such that $0 \in A(x)$ whenever $(v, x - y_0) = 0$ for all $v \in A(x)$.

(3) For the y_0 in (2), if $x_n \rightarrow x$, $v_n \in A(x_n)$, $\{v_n\}$ is bounded and $\lim_{n \rightarrow \infty} (v_n, x_n - y_0) = 0$, then $0 \in A(x)$.

(Remark) If A satisfies (1) and (2), A is called *firmly positive*.

Bruck showed the convergence theorems with respect to a *demi*positive mapping.

定理 (Theorem1) ([1] Bruck, 1974)

Suppose $A : X \rightarrow 2^X$ is *demi*positive and that $x(\cdot) : [0, \infty) \rightarrow X$ is an *absolutely continuous* mapping satisfying

$$\begin{cases} x(t) \in D(A) \text{ for all } t \geq 0, \\ \dot{x}(t) \in -A(x(t)) \text{ for almost all } t > 0, \\ \|x(t)\| \in L^\infty(0, \infty). \end{cases}$$

Then there exists $x^* = w\text{-}\lim_{t \rightarrow \infty} x(t)$ and $x^* \in A^{-1}(0)$.

定理 (Theorem2) ([1] Bruck, 1974) Let $\varphi : X \rightarrow (-\infty, \infty]$ be a proper lower semi-continuous convex even function with a minimum. Then there exists a unique solution $x(\cdot) : [0, \infty) \rightarrow X$, which is absolutely continuous on $[\delta, \infty)$ for all $\delta > 0$, satisfying

$$\begin{cases} x(t) \in D(\partial\varphi) \text{ for all } t > 0, \\ \dot{x}(t) \in -\partial\varphi(x(t)) \text{ for almost all } t > 0, \end{cases}$$

and there exists $x^* = s\text{-}\lim_{t \rightarrow \infty} x(t)$ such that $\varphi(x^*) = \min_{x \in X} \varphi(x)$.

We shall introduce the sufficient conditions of *demi*positivity.

定理 (Theorem3) ([1] Bruck, 1974) A set-valued mapping $A : X \rightarrow 2^X$ is *demi*positive when the following one of (a)–(e) holds:

(a) A is a subdifferential $\partial\varphi$ of a proper lower semi-continuous convex function $\varphi : X \rightarrow [-\infty, \infty)$ with a minimum in X .

(b) A is $I - T$, where I is an identity function and T is a non-expansive mapping with a fixed point.

(c) A is maximal monotone, odd and firmly positive.

(d) A is maximal monotone and $\text{int}A^{-1}(0) \neq \emptyset$.

(e) A is maximal monotone, firmly positive and weakly closed.

(Remark) (1) $A : X \rightarrow 2^X$ is called monotone if $(u - v, x - y) \geq 0$ for any $x, y \in D(A)$ and $u \in A(x), v \in A(y)$.

(2) A is called maximal monotone if it is not properly contained in any other monotone subset of X .

(3) A is said to be weakly closed if $x_n \rightarrow x$, $v_n \rightarrow v$, $v_n \in A(x_n)$ and then $v \in A(x)$.

There are many results of approximations of equilibrium points for Maximal operators ([2], [3], etc.)

2. Solutions of Variational Inequality and Equilibrium points of Differential

Variational Inequality

Let $F : K \rightarrow 2^X$ be a upper semi-continuous set-valued mapping such that $F(x)$ is a non-empty convex compact subset of X for any $x \in K$, where F is said to be upper semi-continuous if for any open set U containing $F(x_0)$ there exists a neighborhood V of x_0 such that $F(V) \subset U$, where $F(V) = \bigcup_{x \in V} F(x)$. We give some definitions for solutions of variational inequalities for F and K .

定義 (Def.) [4] (1) $x^* \in K$ is called a solution of Stampacchia variational inequality $SVI(F, K)$ if there exists $\xi^* \in F(x^*)$ such that $(\xi^*, y - x^*) \geq 0$ for all $y \in K$.

定義 (Def.) (2) $x^* \in K$ is called a solution of Strong Minty variational inequality $SMVI(F, K)$ if for all $y \in K$, $(\eta, y - x^*) \geq 0$ for all $\eta \in F(y)$.

定義 (Def.) (3) $x^* \in K$ is called a solution of Weak Minty variational inequality $WMVI(F, K)$ if for any $y \in K$ there exists $\eta_0 \in F(y)$ such that $(\eta_0, y - x^*) \geq 0$.

F is said to be pseudomonotone if for all $x, y \in K$ there exists $u \in F(x)$ such that $(u, y - x) \geq 0$ then $(v, y - x) \geq 0$ for all $v \in F(y)$. If F is pseudomonotone, the set of solutions of $SVI(F, K)$ coincides with the set of solutions of $SMVI(F, K)$, $WMVI(F, K)$. Results with respect to the convergence theorems of variational inequalities are shown in many approaches ([5], [6], [7]).

Let $T_K(x) = \{v \in X : x + a_n v_n \in K, a_n > 0, a_n \rightarrow 0, v_n \rightarrow v (n \rightarrow \infty)\}$ and let $N_K(x) = \{y \in X : (y, v) \leq 0 \text{ for any } v \in T_K(x)\}$. $T_K(x)$ is called a tangent cone and $N_K(x)$ is called a normal cone. The following differential inclusion is said to be a differential variational inequality $DVI(F, K)$.

$$DVI(F, K) : \begin{cases} x(t) \in K \text{ for all } t \in [0, \infty), \\ \dot{x}(t) \in -(F + N_K)(x(t)) \text{ for a.e. } t \in [0, \infty). \end{cases}$$

And we call the following differential inclusion a projected differential inclusion $PDI(F, K)$.

$$PDI(F, K) : \begin{cases} x(t) \in K \text{ for all } t \in [0, \infty), \\ \dot{x}(t) \in P_{T_K(x)}(-F)(x(t)) \text{ for a.e. } t \in [0, \infty), \end{cases}$$

where $P_{T_K(x)}$ is a projection onto $T_K(x(t))$. It is shown that $x(t)$ is a solution of $DVI(F, K)$ if and only if $x(t)$ is a solution of $PDI(F, K)$.

G.P.Crespi and M.Rocca showed the following theorems ([8]).

定理 (Theorem (G.P.Crespi and M.Rocca, 2004)) Let $x^* \in K$ be an equilibrium point of $DVI(F, K)$ and assume that F is pseudomonotone. Then every solution $x(t)$ of $DVI(F, K)$ satisfies that

$$\|x(t) - x^*\| \leq \|x(s) - x^*\| \quad \text{for } t \geq s.$$

We shall introduce the relation between equilibrium points of $DVI(F, K)$ and solutions of $SVI(F, K)$.

定理 (Theorem) Let $K \subset X$ be a closed convex subset, and let $F : X \rightarrow 2^X$ be an upper semi-continuous mapping with non-empty convex and compact values. Assume F is pseudomonotone. Then, the following (a) and (b) are equivalent:

(a) $x^* \in K$ is an equilibrium point of $DVI(F, K)$. (b) $x^* \in K$ is a solution of $SVI(F, K)$.

There are many results of convergence theorems to solutions of $SVI(F, K)$ by using iterative schemes and also given many results of approximating solutions. We try to study the approximation theory and the iterative methods in order to find an equilibrium point of $DI(F, K)$ with respect to the fixed point theory with good compositions of operator F of a large class of set-valued mappings. ([9])

[References]

- [1] Ronald E. Bruck, Jr. "Asymptotic Convergence of Nonlinear Contraction Semigroups in Hilbert Space," Bull.Ams., 1974
- [2] S.Kamimura and W. Takahashi, "Approximation Solutions of Maximal Monotone Operators in Hilbert Spaces," J. Approx. Theory 106-2(2000), 226-240
- [3] W. Takahashi, "Nonlinear Functional Analysis," Yokohama Publishers, 2002
- [4] D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inequalities and Their Applications," Academic Press, 1980
- [5] U. Mosco, "Convergence of Convex Sets and Solutions of Variational Inequalities," Adv. Math. 3(1969), 510-585
- [6] H. Manaka Tamura, "Regularization of Nonlinear Variational Inequalities with pseudomonotone operators and Mosco-perturbed Domains," Proceedings of NACA, 2001
- [7] J. C. Yao, "Variational Inequalities with generalized monotone operators," Mathematics of Operators Research 19(1994), 691-705
- [8] G. Crespi and M. Rocca, "Minty Variational Inequalities and Monotone Trajectories of Differential Inclusions," Journal inequalities in Pure and Applied Mathematics, vol.5, Issue 2, Article 48, 2004
- [9] H. Manaka Tamura, "A Note on Stević's Iteration Method," J. Math. Anal. Appl. 314(2006), 382-389